On the quantum inverse problem for the closed Toda chain

2004 J. Phys. A: Math. Gen. 37 303

(http://iopscience.iop.org/0305-4470/37/2/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 303–316 PII: S0305-4470(04)63479-2

On the quantum inverse problem for the closed Toda chain

O Babelon[1]

Laboratoire de Physique Théorique et Hautes Energies, Universités Paris VI–Paris VII
(UMR 7589), Boîte 126, Tour 16, 1$^{\text{er}}$ étage, 4 place Jussieu, F-75252, Paris cedex 05, France

Received 14 May 2003, in final form 30 June 2003
Published 15 December 2003
Online at stacks.iop.org/JPhysA/37/303 (DOI: 10.1088/0305-4470/37/2/002)

Abstract

We reconstruct the canonical operators p_i, q_i of the quantum closed Toda chain in terms of Sklyanin's separated variables.

PACS numbers: 02.30.Ik, 05.50.+q, 75.10.Jm

1. Introduction

The theory of classical integrable systems relies on two main ingredients. One is group theory which is used to construct Lax matrices as coadjoint orbits of loop groups, and the second one is complex analysis of the spectral curve, Γ, which is used to effectively solve the models.

In fact, once Γ is given to us, we only need $g = \text{genus}(\Gamma)$ points on it to reconstruct everything. The divisor $\mathcal{D}$ of these g points is called the dynamical divisor. Its role is fundamental. For instance, under an integrable flow, the curve Γ is fixed but the points of $\mathcal{D}$ move on it. The main theorem of integrable systems states that the image of $\mathcal{D}$ by the Abel map, which is a point of the Jacobian of Γ, moves linearly under such flows. Another very important property, which has emerged gradually, is that the coordinates of the points of $\mathcal{D}$ form a set of separated variables in the sense of the Hamilton–Jacobi theory [13, 14].

In quantum theory also, these separated variables, known as Sklyanin's variables, play an important role [1]. It was recently observed that the quantum commuting Hamiltonians had a simple and general expression in terms of the Sklyanin variables [12]. Hence, it becomes natural to set up a quantization procedure of a classical integrable system by using these variables systematically.

In this paper, as an example, we perform this quantization programme in the case of the closed Toda chain. We will be able to reconstruct the original quantum Toda variables in terms of the Sklyanin variables, see equation (23). So, even in this most studied system, the approach seems powerful enough to provide new results.

But before dealing with the specific example of the Toda chain, it is worth recalling a few general facts about the classical theory of integrable systems.

---

[1] Member of CNRS.

Lax matrices built with the help of coadjoint orbits of loop groups lead to spectral curves of the very special form [5, 7, 14]

$$\Gamma : R(\lambda, \mu) \equiv R_0(\lambda, \mu) + \sum_j R_j(\lambda, \mu) H_j = 0 \tag{1}$$

where the H_j are the Poisson commuting Hamiltonians. The coefficients $R_j(\lambda, \mu)$ have a simple geometrical meaning. It turns out that varying the moduli H_i at λ constant, one can show that [5]

$$\delta\mu \, \mathrm{d}\lambda = \text{holomorphic.} \tag{2}$$

Any basis ω_j of holomorphic differentials on Γ can be presented as

$$\omega_j = \frac{N_j(\lambda, \mu)}{\partial_\mu R(\lambda, \mu)} \, \mathrm{d}\lambda = \sigma_j(\lambda, \mu) \, \mathrm{d}\lambda.$$

Since

$$\delta\mu \, \mathrm{d}\lambda = -\sum_j \frac{R_j(\lambda, \mu)}{\partial_\mu R(\lambda, \mu)} \, \mathrm{d}\lambda \, \delta H_j$$

we see that the coefficients $R_j(\lambda, \mu)$ are in fact the numerators $N_j(\lambda, \mu)$ of a basis of holomorphic differentials on Γ. The great virtue of equation (2) is that it implies that there are exactly g-independent Hamiltonians because the space of holomorphic differentials is of dimension g. This is a most welcome fact because the natural candidates for the angle variables are the g angles on the (complex) Jacobian torus, and so we also need g (complex) action variables. This counting argument still holds if we generalize equation (2) as follows:

$$\frac{\delta\mu \, \mathrm{d}\lambda}{f(\lambda, \mu)} = \text{holomorphic} \qquad \text{then} \quad \frac{R_j(\lambda, \mu)}{\partial_\mu R(\lambda, \mu)} = f(\lambda, \mu)\sigma_j(\lambda, \mu). \tag{3}$$

Note that if we consider $\tilde{R}(\lambda, \mu) = h(\lambda, \mu) R(\lambda, \mu)$, where $h(\lambda, \mu)$ does not contain dynamical moduli, we have

$$\frac{\tilde{R}_j}{\partial_\mu \tilde{R}} = \frac{h R_j}{h \partial_\mu R + R \partial_\mu h}$$

so that on Γ the factor h disappears, and the factor f above has an intrinsic meaning.

The g moduli H_i in equation (1) are completely determined if we require that the curve passes through g points $\gamma_k = (\lambda_k, \mu_k)$. Indeed we just have to solve the linear system

$$\sum_{j=1}^{g} R_j(\lambda_k, \mu_k) H_j + R_0(\lambda_k, \mu_k) = 0 \qquad k = 1, \ldots, g \tag{4}$$

whose solution is

$$H = -B^{-1} V \tag{5}$$

where

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_i \\ \vdots \\ H_g \end{pmatrix} \qquad B = \begin{pmatrix} R_1(\lambda_1, \mu_1) & \cdots & R_g(\lambda_1, \mu_1) \\ \vdots & & \vdots \\ R_1(\lambda_i, \mu_i) & \cdots & R_g(\lambda_i, \mu_i) \\ \vdots & & \vdots \\ R_1(\lambda_g, \mu_g) & \cdots & R_g(\lambda_g, \mu_g) \end{pmatrix} \qquad V = \begin{pmatrix} R_0(\lambda_1, \mu_1) \\ \vdots \\ R_0(\lambda_i, \mu_i) \\ \vdots \\ R_0(\lambda_g, \mu_g) \end{pmatrix}.$$

On the $2g$ complex numbers (λ_k, μ_k), we can introduce a non-degenerate Poisson structure

$$\{\lambda_k, \lambda_{k'}\} = 0 \qquad \{\lambda_k, \mu_{k'}\} = p(\lambda_k, \mu_k)\delta_{kk'} \qquad \{\mu_k, \mu_{k'}\} = 0. \tag{6}$$

We do not need to specify the function $p(\lambda, \mu)$ for the moment. In the case of a spectral curve of the Lax matrix with a linear bracket in the r-matrix language, it is known that generically $p(\lambda_k, \mu_k) = 1$. In the case of the quadratic Sklyanin bracket with rational r-matrix, such as the Toda chain below, we have $p(\lambda_k, \mu_k) = \mu_k$. In the case of a trigonometric r-matrix we have rather $p(\lambda_k, \mu_k) = \lambda_k\mu_k$ [9]. By a simple calculation, we prove [10, 12]:

Proposition 1. *For any function $p(\lambda, \mu)$ in equation (6), the Hamiltonians defined by equation (5) are in involution*

$$\{H_i, H_j\} = 0.$$

The H_i therefore define integrable flows on the $2g$-dimensional phase space equation (6).

There is an interesting relation between the functions $p(\lambda, \mu)$ entering the Poisson bracket, equation (6), and the function $f(\lambda, \mu)$ in equation (3). We define the angles as the images of the divisor (λ_k, μ_k) by the Abel map:

$$\theta_j = \sum_k \int^{\lambda_k} \sigma_j(\lambda, \mu)\, \mathrm{d}\lambda.$$

This defines a point on the Jacobian of Γ.

Proposition 2. *Under the above map, the flows generated by the Hamiltonians H_i are linear on the Jacobian if and only if $f(\lambda, \mu) = p(\lambda, \mu)$.*

Proof. We want to show that the velocities $\partial_{t_i}\theta_j$ are constant, or

$$\partial_{t_i}\theta_j = \sum_k \partial_{t_i}\lambda_k\sigma_j(\lambda_k, \mu_k) = C_{ij}^{ste}.$$

Indeed, one has

$$\begin{aligned}\partial_{t_i}\lambda_k &= \{H_i, \lambda_k\} = -\left\{B_{il}^{-1}V_l, \lambda_k\right\} \\ &= B_{ir}^{-1}\{B_{rs}, \lambda_k\}B_{sl}^{-1}V_l - B_{il}^{-1}\{V_l, \lambda_k\} \\ &= -B_{ik}^{-1}[\{B_{ks}, \lambda_k\}H_s + \{V_k, \lambda_k\}]\end{aligned}$$

where, in the last line, we used the separated structure of the matrix B and the vector V. Explicitly

$$\begin{aligned}\partial_{t_i}\lambda_k B_{kj} &= B_{ik}^{-1}B_{kj}[\partial_\mu R_s(\lambda_k, \mu_k)H_s + \partial_\mu R_0(\lambda_k, \mu_k)]p(\lambda_k, \mu_k) \\ &= B_{ik}^{-1}B_{kj}\partial_\mu R(\lambda_k, \mu_k)p(\lambda_k, \mu_k).\end{aligned}$$

It follows that

$$\partial_{t_i}\lambda_k \frac{R_j(\lambda_k, \mu_k)}{p(\lambda_k, \mu_k)\partial_\mu R(\lambda_k, \mu_k)} = B_{ik}^{-1}B_{kj}$$

summing over k gives

$$\sum_k \partial_{t_i}\lambda_k \frac{f(\lambda_k, \mu_k)}{p(\lambda_k, \mu_k)}\sigma_j(\lambda_k, \mu_k) = \delta_{ij} \tag{7}$$

which reduces to what we had to prove when $f(\lambda, \mu) = p(\lambda, \mu)$. □

Hence equations (1), (3) do provide us integrable systems and their solutions. The main question in this approach is to go back from the separated variables (λ_k, μ_k) to the 'original' variables, i.e. the ones entering the Lax matrix elements. A Lax matrix is a matrix $L(\lambda)$ depending rationally on λ such that

$$R(\lambda, \mu) = \det(L(\lambda) - \mu).$$

A general strategy to construct it is as follows [3, 14]. First, we determine the size of the matrix $L(\lambda)$, by looking at the curve Γ as a covering of the λ-plane $(\lambda, \mu) \to \lambda$. The dimension of the matrix is just the number of sheets of this covering. Let us assume that the curve Γ is not ramified at $\lambda = \infty$. Call Q_i $(\lambda = \infty, a_i), i = 1, \dots, N$ the point above $\lambda = \infty$. We can normalize $L(\lambda) = \mathrm{Diag}(a_1, a_2, \dots, a_N) + 0(1/\lambda)$ at $\lambda = \infty$. To each point $P(\lambda, \mu)$ of the curve Γ, not a branch point of the covering $(\lambda, \mu) \to \lambda$, one can attach the one-dimensional eigenspace of $L(\lambda)$ corresponding to the eigenvalue μ. One can show that this extends to an analytic line bundle on Γ with the Chern class $g + N - 1$. The eigenvector $\Psi(P)$ at $P \in \Gamma$ can be presented as

$$\Psi(P) = \begin{pmatrix} 1 \\ \psi_2(P) \\ \vdots, \\ \psi_N(P) \end{pmatrix} \qquad [\psi_i] = Q_1 - Q_i - \mathcal{D}.$$

The function $\psi_i, i = 2, \dots, N-1$ has a zero at Q_1, a pole at Q_i and g poles at a divisor $\mathcal{D}$ at finite distance. By the Riemann–Roch theorem, this function exists and is unique for $\mathcal{D}$ generic. So apart from the $N - 1$ poles at infinity which are fixed, all the important information is contained in the dynamical divisor $\mathcal{D}$. We identify $\mathcal{D}$ with the divisor of the g points $\gamma_k = (\lambda_k, \mu_k)$ above and construct the corresponding vector $\Psi(P)$. Once this is done, we consider the N points P_i above λ, and build the matrices

$$\widehat{\Psi} = (\Psi(P_1), \dots, \Psi(P_N)) \qquad \widehat{\mu} = \mathrm{Diag}(\mu(P_1), \dots, \mu(P_N)).$$

The matrix $L(\lambda)$ is given by

$$L(\lambda) = \widehat{\Psi}\widehat{\mu}\widehat{\Psi}^{-1}.$$

This is independent of the order of the points P_i, and is a rational function of λ.

This method gives a way, in principle, to reconstruct the Lax matrix starting only from the spectral curve and the dynamical divisor on it, hence returning to the original variables. Of course, in concrete examples, some of the genericity assumptions made here may have to be modified, or shortcuts may be available, but the general ideas remain the same.

2. The classical Toda chain

The closed Toda chain is defined by the Hamiltonian [2]

$$H = \sum_{i=1}^{n+1} \frac{1}{2} p_i^2 + \mathrm{e}^{q_{i+1} - q_i} \tag{8}$$

where we assume that $q_{n+2} \equiv q_1$, and Poisson bracket

$$\{q_i, q_j\} = 0 \qquad \{p_i, q_j\} = \delta_{ij} \qquad \{p_i, p_j\} = 0.$$

This is an integrable system. We associate with it the Lax matrix as follows. Consider the 2×2 matrices

$$T_j(\lambda) = \begin{pmatrix} \lambda + p_j & -\mathrm{e}^{q_j} \\ \mathrm{e}^{-q_j} & 0 \end{pmatrix}$$

and construct

$$T(\lambda) = T_1(\lambda) \cdots T_2(\lambda) T_{n+1}(\lambda). \tag{9}$$

We can write

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \qquad A(\lambda)D(\lambda) - B(\lambda)C(\lambda) = 1 \tag{10}$$

where $A(\lambda)$ is a polynomial of degree $n + 1$, $D(\lambda)$ is of degree $n - 1$ and $B(\lambda)$, $C(\lambda)$ are of degree n. The spectral curve is defined as usual

$$\det(T(\lambda) - \mu) = 0 \equiv \mu + \mu^{-1} - t(\lambda) = 0 \tag{11}$$

where

$$t(\lambda) = A(\lambda) + D(\lambda) = \lambda^{n+1} + \sum_{j=0}^{n} \lambda^j H_j \qquad H_n = P \qquad H_{n-1} = \frac{1}{2}P^2 - H$$

where $P = \sum_i p_i$, and H is given by equation (8). The $n + 1$ quantities H_j are conserved. The curve equation (11) is hyperelliptic. It can be written as

$$s^2 = t^2(\lambda) - 4 \qquad \text{with} \qquad s = 2\mu - t(\lambda) = \mu - \mu^{-1}. \tag{12}$$

The polynomial $t^2(\lambda)$ being of degree $2(n + 1)$, the genus of the curve is $g = n$. The number of dynamical moduli is $g = n$ in the centre of mass frame $P = 0$. In the following we therefore always consider the system reduced by the translational symmetry. We have

$$\frac{\delta\mu}{\mu}\,\mathrm{d}\lambda = \frac{\delta t(\lambda)}{\mu - \mu^{-1}}\,\mathrm{d}\lambda = \frac{\delta t(\lambda)}{s}\,\mathrm{d}\lambda = \text{holomorphic}. \tag{13}$$

Asking that the curve equation (11) passes through the n points (λ_i, μ_i), we get n equations

$$t(\lambda_i) = \mu_i + \mu_i^{-1}.$$

Their solution for the n Hamiltonians H_i may be cast conveniently in the form of Lagrange interpolation formula:

$$t(\lambda) = t^{(0)}(\lambda) + t^{(1)}(\lambda) \tag{14}$$

where

$$t^{(0)}(\lambda) = \left(\lambda + \sum_i \lambda_i\right) \prod_{i=1}^{n} (\lambda - \lambda_i) \qquad t^{(1)}(\lambda) = \sum_i \prod_{j \neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j} \big(\mu_i + \mu_i^{-1}\big). \tag{15}$$

The polynomial $t^{(0)}(\lambda)$ is of degree $n + 1$, vanishes for $\lambda = \lambda_i$ and has no λ^n term.

We define the Poisson bracket of the separated variables as (in agreement with equations (6), (13))

$$\{\lambda_k, \lambda_{k'}\} = 0 \qquad \{\mu_k, \lambda_{k'}\} = \mu_k \delta_{kk'} \qquad \{\mu_k, \mu_{k'}\} = 0.$$

By the general result of [10, 12] the Hamiltonians H_i obtained as coefficients of the polynomial $t(\lambda)$ in equation (15) are in involution. Note that the above Poisson bracket is the one matching the condition (13) and leads to flows linearizing on the Jacobian of the spectral curve equation (11).

To proceed, we reconstruct the Lax matrix. The curve equation (11) is a two-sheeted cover of the λ-plane. For a 2 × 2 matrix of the form of equation (10) the eigenvector is simple

$$(T(\lambda) - \mu)\Psi = 0 \qquad \Psi = \begin{pmatrix} 1 \\ \psi_2 \end{pmatrix} \qquad \psi_2 = -\frac{A(\lambda) - \mu}{B(\lambda)}.$$

The poles of ψ_2 at finite distance are above the zeroes λ_i of $B(\lambda) = 0$ which is a polynomial of degree n. The two points above λ_i are $\mu_i^+ = A(\lambda_i)$, $\mu_i^- = D(\lambda_i)$ so that ψ_2 has a pole only on the second point. The points of the dynamical divisor are therefore

$$(\lambda_i, D(\lambda_i)) \qquad B(\lambda_i) = 0.$$

Given the points of the dynamical divisor, we reconstruct $A(\lambda)$ and $B(\lambda)$:

$$B(\lambda) = b_0 \prod_{i=1}^{n} (\lambda - \lambda_i)$$

$$A(\lambda) = \left(\lambda + \sum_{i=1}^{n} \lambda_i\right) \prod_{i=1}^{n} (\lambda - \lambda_i) + \sum_i \mu_i \frac{\prod_{j\neq i}^{n} (\lambda - \lambda_j)}{\prod_{j\neq i}^{n} (\lambda_i - \lambda_j)}.$$

Knowing $A(\lambda)$ and $B(\lambda)$ we reconstruct $C(\lambda)$ and $D(\lambda)$ by the trace and determinant conditions. These formulae were the basis of Sklyanin's work [1] and of Smirnov's work [6, 9] (with a different Poisson structure).

Reconstructing the original degrees of freedom of the Toda chain, however, is equivalent to reconstructing the $(n + 1) \times (n + 1)$ Lax matrix:

$$L(\mu) = \sum_i p_i E_{ii} + \sum_{i=1}^{n} \mathrm{e}^{\frac{1}{2}(q_{i+1}-q_i)} (E_{i,i+1} + E_{i+1,i}) + \mathrm{e}^{\frac{1}{2}(q_1-q_{n+1})} (\mu E_{n+1,1} + \mu^{-1} E_{1,n+1})$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. This matrix is such that

$$\det(L(\mu) - \lambda) = \mu + \mu^{-1} - A(\lambda) - D(\lambda).$$

Since $L(\mu)$ is of size $(n + 1) \times (n + 1)$, we look at the spectral curve equation (11) as a $(n + 1)$-sheeted cover of the μ-plane. When $\lambda = \infty$, we have two points P^+ and P^- corresponding to $\mu = \infty$ and $\mu = 0$ respectively,

$$P^+ : \mu = \lambda^{n+1}(1 + O(\lambda^{-2})) \qquad P^- : \mu = \lambda^{-n-1}(1 + O(\lambda^{-2})).$$

According to, e.g., [3, 14], the eigenvectors of $L(\mu)$ are easy to construct. Set

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \mu \end{pmatrix}$$

where we have normalized the last component to be μ. The meromorphic functions ψ_i have poles at the dynamical divisor; moreover

$$\psi_i = \mathrm{e}^{\frac{q_i-q_{n+1}}{2}} \lambda^i (1 + O(\lambda^{-1}) \qquad \text{near } P^+$$

$$\psi_i = \mathrm{e}^{-\frac{q_i-q_{n+1}}{2}} \lambda^{-i} (1 + O(\lambda^{-1}) \qquad \text{near } P^-.$$

These properties determine the functions ψ_i uniquely. Being meromorphic functions on a hyperelliptic curve, we can write

$$\psi_i = \frac{Q^{(i)}(\lambda) + \mu R^{(i)}(\lambda)}{\prod_{j=1}^{n} (\lambda - \lambda_j)}$$

where $Q^{(i)}$ and $R^{(i)}$ are polynomials. We want the poles to be at (λ_j, μ_j) only so that the numerator should vanish at the points $\left(\lambda_j, \mu_j^{-1}\right)$. This gives n conditions

$$Q^{(i)}(\lambda_j) + \mu_j^{-1} R^{(i)}(\lambda_j) = 0 \qquad j = 1, \ldots, n. \tag{16}$$

To have a pole of order i at P^+ and a zero of order i at P^-, we choose

$$\text{degree } Q^{(i)} = n - i \qquad \text{degree } R^{(i)} = i - 1.$$

These two polynomials depend altogether on $n + 1$ coefficients. They are determined by imposing the n conditions of equation (16) and requiring that the normalization coefficients are inverse to each other at $P^{\pm}$. We set

$$\begin{aligned} Q^{(i)}(\lambda) &= Q_0^{(i)} + Q_1^{(i)}\lambda + \cdots + Q_{n-i}^{(i)}\lambda^{n-i} \\ R^{(i)}(\lambda) &= R_0^{(i)} + R_1^{(i)}\lambda + \cdots + R_{i-1}^{(i)}\lambda^{i-1}. \end{aligned}$$

Moreover, since $\psi_{n+1} = \mu$, we have to define

$$Q^{(n+1)}(\lambda) = 0 \qquad R^{(n+1)}(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j)$$

then equations (16) become

$$\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{i-1} & \mu_1 & \mu_1\lambda_1 & \cdots & \mu_1\lambda_1^{n-i-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_j & \cdots & \lambda_j^{i-1} & \mu_j & \mu_j\lambda_j & \cdots & \mu_j\lambda_j^{n-i-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{i-1} & \mu_n & \mu_n\lambda_n & \cdots & \mu_n\lambda_n^{n-i-1} \end{pmatrix} \begin{pmatrix} R_0^{(i)} \\ \vdots \\ R_{i-1}^{(i)} \\ Q_0^{(i)} \\ \vdots \\ Q_{n-i-1}^{(i)} \end{pmatrix} = -Q_{n-i}^{(i)} \begin{pmatrix} \mu_1\lambda_1^{n-i} \\ \vdots \\ \mu_j\lambda_j^{n-i} \\ \vdots \\ \mu_n\lambda_n^{n-i} \end{pmatrix}$$

or, with obvious notation; $M^{(i)}W^{(i)} = -Q_{n-i}^{(i)}V^{(i)}$ and therefore $W^{(i)} = -Q_{n-i}^{(i)}M^{(i)-1}V^{(i)}$. In particular

$$R_{k-1}^{(i)} = -Q_{n-i}^{(i)}\frac{\Delta_k^{(i)}}{\Delta^{(i)}}$$

where $\Delta^{(i)} = \det M^{(i)}$, and $\Delta_k^{(i)}$ is the determinant of the matrix obtained from $M^{(i)}$ by replacing column k by $V^{(i)}$. Finally, one has to impose that the leading coefficients at $P_{\pm}$ are inverse to each other: $R_{i-1}^{(i)} = \left(Q_{n-i}^{(i)}\right)^{-1}$. This gives

$$\left(Q_{n-i}^{(i)}\right)^{-2} = \mathrm{e}^{q_i - q_{n+1}} = -\frac{\Delta_i^{(i)}}{\Delta^{(i)}}.$$

To reconstruct the momenta, we follow [14] again. Expand

$$\psi_i = \mathrm{e}^{\frac{q_i - q_{n+1}}{2}}\lambda^i(1 - \xi_i\lambda^{-1} + \cdots) \qquad \text{near } P^+$$

then $p_i = \xi_{i+1} - \xi_i$. We find at once

$$\xi_i = -\sum_{j=1}^{n}\lambda_j - \frac{R_{i-2}^{(i)}}{R_{i-1}^{(i)}}$$

hence

$$p_i = \frac{\Delta_{i-1}^{(i)}}{\Delta_i^{(i)}} - \frac{\Delta_i^{(i+1)}}{\Delta_{i+1}^{(i+1)}}$$

which we complement with the boundary terms

$$p_1 = -\frac{\Delta_1^{(2)}}{\Delta_2^{(2)}} \qquad p_n = \frac{\Delta_{n-1}^{(n)}}{\Delta_n^{(n)}} + \sum_{j=1}^{n}\lambda_j \qquad p_{n+1} = -\sum_{j=1}^{n}p_j.$$

We now give more explicit formulae for the determinants entering the above expressions. We call $[k]$ a subset of cardinality k of $(1, 2, \ldots, n)$:

$$[k] = (i_1, i_2, \ldots, i_k).$$

We write $\sum_{[k]}$ for the sum over all such sets. Define

$$S_{[k]} = \prod_{i\in[k]} \prod_{j\neq[k]} \frac{1}{(\lambda_i - \lambda_j)} \tag{17}$$

and

$$\mu_{[k]} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}$$

then, we have

$$X^{(k)} \equiv \frac{\Delta^{(n-k)}}{\Delta^{(n)}} = \sum_{[k]} S_{[k]}\mu_{[k]} \tag{18}$$

$$Y^{(k)} \equiv \frac{\Delta^{(n-k)}_{n-k-1}}{\Delta^{(n)}} = \sum_{[k]} S_{[k]} \left(\sum_{i\notin[k]} \lambda_i\right) \mu_{[k]}. \tag{19}$$

We have (note that $\Delta^{(i)}_i = (-1)^{n-i}\Delta^{(i-1)}$)

$$\mathrm{e}^{q_i - q_{n+1}} = \frac{X^{(n-i+1)}}{X^{(n-i)}} \qquad p_i = \frac{Y^{(n-i+1)}}{X^{(n-i+1)}} - \frac{Y^{(n-i)}}{X^{(n-i)}}. \tag{20}$$

It remains to check that the Poisson bracket between p_i, q_i is canonical. This easily follows from

Proposition 3.

$$\{X^{(k)}, X^{(l)}\} = 0$$
$$\{X^{(k)}, Y^{(l)}\} = (k-l)\theta(k-l)X^{(k)}X^{(l)}$$
$$\{Y^{(k)}, Y^{(l)}\} = (k-l)(\theta(k-l)Y^{(k)}X^{(l)} + \theta(l-k)X^{(k)}Y^{(l)})$$

where $\theta(k-l) = 1$ *if* $k > l, 0$ *otherwise.*

Instead of proving these relations directly, it is more convenient to use the quantity $Z^{(k)}$ defined by

$$Y^{(k)} = \left(\sum_{i=1}^{n} \lambda_i\right) X^{(k)} - Z^{(k)} \qquad Z^{(k)} = \sum_{[k]} S_{[k]} \left(\sum_{i\in[k]} \lambda_i\right) \mu_{[k]}. \tag{21}$$

Since $\left\{\sum_{i=1}^n \lambda_i, X^{(k)}\right\} = -kX^{(k)}$, $\left\{\sum_{i=1}^n \lambda_i, Z^{(k)}\right\} = -kZ^{(k)}$ we have to show that

Proposition 4.

$$\{X^{(k)}, X^{(l)}\} = 0$$
$$\{X^{(k)}, Z^{(l)}\} = (l\theta(k-l) + k\theta(l-k))X^{(k)}X^{(l)}$$
$$\{Z^{(k)}, Z^{(l)}\} = (l\theta(k-l) + k\theta(l-k))(Z^{(k)}X^{(l)} - X^{(k)}Z^{(l)}).$$

Proof. Take the semiclassical limit of the quantum formulae below. □

Equations (20) and the above proposition provide a complete solution to the problem of expressing the original Toda variables p_i, q_i in terms of the separated variables in the classical case. We now turn to quantum theory.

3. The quantum Toda chain

In the quantum case, analysis on the Riemann surfaces is not available. So, we try to quantize directly the relevant classical formulae.

Quantum commutation relations are defined directly on the separated variables.

$$[\lambda_k, \lambda_{k'}] = 0 \qquad \mu_k\lambda_{k'} = (\lambda_{k'} + \mathrm{i}\hbar\delta_{kk'})\mu_k \equiv (t_k\lambda_{k'})\mu_k \qquad [\mu_k, \mu_{k'}] = 0.$$

As shown in [12], the formulae (15) for the quantum Hamiltonians remain valid at the quantum level (with the μ_i written on the right) and they are all commuting.

The new result of this paper concerns the variables q_i, p_i of the Toda chain. We show that the classical formulae of equations (20) can also be straightforwardly quantized.

As a first step, we quantize the operators $X^{(k)}$ and $Z^{(k)}$. We define them by the same formulae as in the classical case of equations (18), (21), but now it is important to write the μ_i to the right. We have

Proposition 5.

$$\begin{aligned}
&[X^{(k)}, X^{(l)}] = 0\\
&[X^{(k)}, Z^{(l)}] = \mathrm{i}\hbar(k\theta(l-k) + l\theta(k-l))X^{(k)}X^{(l)}\\
&[Z^{(k)}, Z^{(l)}] = \mathrm{i}\hbar(k\theta(l-k) + l\theta(k-l))(Z^{(k)}X^{(l)} - Z^{(l)}X^{(k)}).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&[X^{(k)}, X^{(l)}] = \sum_{[k],[l]} (S_{[k]}t_{[k]}S_{[l]} - S_{[l]}t_{[l]}S_{[k]})\mu_{[k]}\mu_{[l]}\\
&[X^{(k)}, Z^{(l)}] = \sum_{[k],[l]} (S_{[k]}t_{[k]}(\lambda_{[l]}S_{[l]}) - \lambda_{[l]}S_{[l]}(t_{[l]}S_{[k]}))\mu_{[k]}\mu_{[l]}\\
&[Z^{(k)}, Z^{(l)}] = \sum_{[k],[l]} (\lambda_{[k]}S_{[k]}(t_{[k]}\lambda_{[l]}S_{[l]}) - \lambda_{[l]}S_{[l]}(t_{[l]}\lambda_{[k]}S_{[k]}))\mu_{[k]}\mu_{[l]}
\end{aligned}$$

where we denoted

$$\lambda_{[k]} = \sum_{i\in[k]} \lambda_i.$$

We set

$$[k] = [k'] + [m'] \qquad [l] = [l'] + [m'] \qquad [k'] \cap [l'] = \emptyset. \tag{22}$$

We have

$$\begin{aligned}
&\sum_{[k],[l]} ((\lambda_{[k]})^a S_{[k]}(t_{[k]}(\lambda_{[l]})^b S_{[l]}) - (\lambda_{[l]})^b S_{[l]}(t_{[l]}(\lambda_{[k]})^a S_{[k]}) = \sum_{[k],[l]} (-1)^{k'l'}\\
&\qquad \times \prod_{\substack{i\in[k'+l'+m']\\ j\notin[k'+l'+m']}} \frac{1}{\lambda_i - \lambda_j} \prod_{\substack{i\in[m']\\ j\in[k'+l']}} \frac{1}{\lambda_i - \lambda_j} \prod_{\substack{i\in[m']\\ j\notin[k'+l'+m']}} \frac{1}{\lambda_i - \lambda_j + \mathrm{i}\hbar}\\
&\qquad \times \prod_{\substack{i\in[k']\\ j\in[l']}} \frac{1}{\lambda_i - \lambda_j} \left((\lambda_{[k]})^a (\lambda_{[l]} + \mathrm{i}\hbar m')^b \prod_{\substack{i\in[k']\\ j\in[l']}} \frac{1}{\lambda_i - \lambda_j + \mathrm{i}\hbar} \right.\\
&\qquad \left. - (\lambda_{[l]})^b (\lambda_{[k]} + \mathrm{i}\hbar m')^a \prod_{\substack{i\in[k']\\ j\in[l']}} \frac{1}{\lambda_i - \lambda_j - \mathrm{i}\hbar} \right).
\end{aligned}$$

The coefficient on the second line only depends on $[k'+l']$. Hence we can split the sum

$$\sum_{[k],[l]} = \sum_{[m'],[k'+l']} \sum_{[k'],[l']}$$

and the last sum goes straight to the last line.

If $a = 0, b = 0$, that is in the calculation of $[X^{(k)}, X^{(l)}]$, the last sum vanishes by lemma 2.

If $a = 0, b = 1$, that is in the calculation of $[X^{(k)}, Z^{(l)}]$, we set

$$\lambda_{[l]} + \mathrm{i}\hbar m' = \lambda_{[k'+l'+m']} - \lambda_{[k']} + \mathrm{i}\hbar k - \mathrm{i}\hbar k'$$

$$\lambda_{[l]} = \lambda_{[k'+l'+m']} - (\lambda_{[k']} - \mathrm{i}\hbar k') - \mathrm{i}\hbar k'.$$

By lemma 2 applied with $a = 0, 1$, only the $\mathrm{i}\hbar k$ term contributes in the last sum. This term is exactly equal to

$$[X^{(k)}, Z^{(l)}] = \mathrm{i}\hbar k X^{(k)} X^{(l)} \qquad k < l.$$

If $k > l$, we write this time

$$\lambda_{[l]} + \mathrm{i}\hbar m' = (\lambda_{[l']} - \mathrm{i}\hbar l') + \lambda_{[m']} + \mathrm{i}\hbar l \qquad \lambda_{[l]} = \lambda_{[l']} + \lambda_{[m']}$$

and this time only the $\mathrm{i}\hbar l$ term contributes. Hence

$$[X^{(k)}, Z^{(l)}] = \mathrm{i}\hbar l X^{(k)} X^{(l)}.$$

If $a = 1, b = 1$, that is in the calculation of $[Z^{(k)}, Z^{(l)}]$, we set (assuming $k < l$):

$$\lambda_{[k]}(\lambda_{[l]} + m'\mathrm{i}\hbar) = \mathrm{i}\hbar k \lambda_{[k]} - (\lambda_{[k']})^2 + (\lambda_{[k'+l']} - \mathrm{i}\hbar k')\lambda_{[k']} + \lambda_{[m']}(\lambda_{[k'+l'+m']} - \mathrm{i}\hbar k')$$

$$\begin{aligned}\lambda_{[l]}(\lambda_{[k]} + m'\mathrm{i}\hbar) = {} & \mathrm{i}\hbar k \lambda_{[l]} - (\lambda_{[k']} - \mathrm{i}\hbar k')^2 + (\lambda_{[k'+l']} - \mathrm{i}\hbar k')(\lambda_{[k']} - \mathrm{i}\hbar k') \\ & + \lambda_{[m']}(\lambda_{[k'+l'+m']} - \mathrm{i}\hbar k').\end{aligned}$$

By lemma 2 only the terms $\mathrm{i}\hbar k \lambda_{[k]}$ and $\mathrm{i}\hbar k \lambda_{[l]}$ contribute. Hence

$$[Z^{(k)}, Z^{(l)}] = \mathrm{i}\hbar k (Z^{(k)} X^{(l)} - Z^{(l)} X^{(k)}) \qquad k < l.$$

□

It is now simple to write the commutation relations with $Y^{(k)}$ defined in equation (19).

Proposition 6.

$$[X^{(k)}, X^{(l)}] = 0$$

$$[X^{(k)}, Y^{(l)}] = \mathrm{i}\hbar (k-l)\theta(k-l) X^{(k)} X^{(l)}$$

$$[Y^{(k)}, Y^{(l)}] = \mathrm{i}\hbar (k-l)[\theta(k-l) Y^{(k)} X^{(l)} + \theta(l-k) Y^{(l)} X^{(k)}].$$

We define the quantum Toda variables as in the classical case.

Proposition 7. *Let us define the quantum Toda operators as*

$$\mathrm{e}^{q_i - q_{n+1}} = \frac{X^{(n-i+1)}}{X^{(n-i)}} \qquad p_i = \frac{Y^{(n-i+1)}}{X^{(n-i+1)}} - \frac{Y^{(n-i)}}{X^{(n-i)}}. \tag{23}$$

Then, we have

$$[\mathrm{e}^{q_i}, \mathrm{e}^{q_j}] = 0 \qquad [\mathrm{e}^{q_i}, p_j] = -\mathrm{i}\hbar \delta_{ij}\, \mathrm{e}^{q_i} \qquad [p_i, p_j] = 0. \tag{24}$$

Proof. Note that there is no ordering ambiguity in the expressions (23). The proof of the canonical commutation relations relies on

$$\left[\frac{1}{X^{(k)}}, Y^{(l)}\right] = -\mathrm{i}\hbar (k-l)\theta(k-l) \frac{X^{(l)}}{X^{(k)}}$$

which implies in turn

$$\left[\frac{Y^{(k)}}{X^{(k)}}, \frac{Y^{(l)}}{X^{(l)}}\right] = 0.$$

□

Equations (23), (24) constitute the main result of this paper. It is important to check the reality of our operators. The conjugation operation on the variables λ_k, μ_k was given by Sklyanin:

$$\lambda_k^* = \lambda_k \qquad \mu_k^* = \prod_{j\neq k} \frac{\lambda_k - \lambda_j + \mathrm{i}\hbar}{\lambda_k - \lambda_j}\mu_k.$$

This conjugation rule is found by requiring that the Hamiltonians H_j be self-conjugate. It is a simple exercise to check that the operators $X^{(k)}, Y^{(k)}$ are self-conjugate, and therefore so are p_i, q_i.

4. Conclusion

Equations (23) open up the possibility of computing the matrix elements of these operators between the eigenstates of the Hamiltonians H_i, see [6, 8]. The existing techniques should be sufficient to handle operators polynomial in μ_i, such as

$$\mathrm{e}^{q_n - q_{n+1}}\,\mathrm{e}^{q_{n-1}-q_{n+1}} \cdots \mathrm{e}^{q_{n-k+1}-q_{n+1}} = X^{(k)}.$$

For the operators p_i, e^{q_i} themselves, however, we will have to learn how to treat ratios of such polynomial operators. This is an important issue since for non-hyperelliptic curves this situation seems unavoidable [11].

Acknowledgments

I thank F Smirnov and D Talalaev for discussions, and D Arnaudon, J Avan, L Frappat, E Ragoucy and P Sorba for their kind invitation to the conference RAQIS03.

Appendix

We prove some combinatorial identities which are used in the computation of the commutators of the operators $X^{(k)}, Z^{(k)}$.

Lemma 1.

$$\sum_{i=1}^{n}\prod_{j\neq i}\frac{1}{(\lambda_i - \lambda_j)}\left(\lambda_i^a \prod_{j\neq i}\frac{1}{(\lambda_i - \lambda_j + \mathrm{i}\hbar)} - (\lambda_i - \mathrm{i}\hbar)^a \prod_{j\neq i}\frac{1}{(\lambda_i - \lambda_j - \mathrm{i}\hbar)}\right) = 0$$

for $a \leqslant 2n$.

Proof. Consider

$$Q(z) = z^a \prod_i \frac{1}{z - \lambda_i} \prod_i \frac{1}{z - \lambda_i + \mathrm{i}\hbar}.$$

The residue at the pole $z = \lambda_i$ reads

$$\frac{1}{\mathrm{i}\hbar}\lambda_i^a \prod_{i\neq j}\frac{1}{\lambda_i - \lambda_j}\prod_{i\neq j}\frac{1}{\lambda_i - \lambda_j + \mathrm{i}\hbar}$$

while the residue at $z = \lambda_i - \mathrm{i}\hbar$ reads

$$-\frac{1}{\mathrm{i}\hbar}(\lambda_i - \mathrm{i}\hbar)^a \prod_{i\neq j} \frac{1}{\lambda_i - \lambda_j - \mathrm{i}\hbar} \prod_{i\neq j} \frac{1}{\lambda_i - \lambda_j}.$$

Hence our expression is the sum of the residues of $Q(z)$ at finite distance, which vanishes if $a \leqslant 2n$. $\square$

Lemma 2. *Suppose* $k < n/2$. *Then*

$$\sum_{[k]} \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j)} \left((\lambda_{[k]})^a \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j + \mathrm{i}\hbar)} - (\lambda_{[k]} - \mathrm{i}\hbar k)^a \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j - \mathrm{i}\hbar)} \right) = 0$$

for $0 \leqslant a \leqslant 2(n - 2k + 1)$. *By symmetry, if* $k > n/2$, *we have*

$$\sum_{[k]} \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j)} \left((\lambda_{[n-k]} - \mathrm{i}\hbar(n-k))^a \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j + \mathrm{i}\hbar)} \right.$$

$$\left. - (\lambda_{[n-k]})^a \prod_{\substack{i\in[k]\\ j\notin[k]}} \frac{1}{(\lambda_i - \lambda_j - \mathrm{i}\hbar)} \right) = 0$$

for $0 \leqslant a \leqslant 2(2k - n + 1)$.

Proof. Consider this expression as a function of λ_1. It tends to zero at ∞, and it has poles at the other λ_j, $\lambda_j \pm \mathrm{i}\hbar$. Consider λ_2. We have two contributions corresponding to $\lambda_1 \in [k]$, $\lambda_2 \notin [k]$ and $\lambda_1 \notin [k]$, $\lambda_2 \in [k]$. We denote by $[n']$ the subset of $[n]$ where λ_1 and λ_2 have been removed, by $[k']$ a subset of $[n']$ of cardinality $k-1$ and by $[l']$ the complementary subset in $[n']$. The two contributions can be written, respectively,

$$A = \frac{1}{\lambda_1 - \lambda_2} P_{[l']}(\lambda_1) P_{[k']}(\lambda_2) {\prod}'_0 \left(\frac{(\lambda_1 + \lambda_{[k']})^a}{\lambda_1 - \lambda_2 + \mathrm{i}\hbar} P_{[l']}(\lambda_1 + \mathrm{i}\hbar) P_{[k']}(\lambda_2 - \mathrm{i}\hbar) {\prod}'_+ \right.$$
$$\left. - \frac{(\lambda_1 + \lambda_{[k']} - k\mathrm{i}\hbar)^a}{\lambda_1 - \lambda_2 - \mathrm{i}\hbar} P_{[l']}(\lambda_1 - \mathrm{i}\hbar) P_{[k']}(\lambda_2 + \mathrm{i}\hbar) {\prod}'_- \right)$$

$$B = \frac{1}{\lambda_2 - \lambda_1} P_{[l']}(\lambda_2) P_{[k']}(\lambda_1) {\prod}'_0 \left(\frac{(\lambda_2 + \lambda_{[k']})^a}{\lambda_2 - \lambda_1 + \mathrm{i}\hbar} P_{[l']}(\lambda_2 + \mathrm{i}\hbar) P_{[k']}(\lambda_1 - \mathrm{i}\hbar) {\prod}'_+ \right.$$
$$\left. - \frac{(\lambda_2 + \lambda_{[k']} - k\mathrm{i}\hbar)^a}{\lambda_2 - \lambda_1 - \mathrm{i}\hbar} P_{[l']}(\lambda_2 - \mathrm{i}\hbar) P_{[k']}(\lambda_1 + \mathrm{i}\hbar) {\prod}'_- \right)$$

where

$$P_{[l']}(\lambda) = \prod_{j\in[l']} \frac{1}{\lambda - \lambda_j}$$

and

$${\prod}'_\sigma = \prod_{\substack{i\in[k']\\ j\in[l']}} \frac{1}{(\lambda_i - \lambda_j + \sigma\mathrm{i}\hbar)} \qquad \sigma = 0, \pm.$$

Consider the pole at $\lambda_1 = \lambda_2$. Set $\lambda_1 = \lambda_2 + \epsilon$.

$$A = \frac{1}{\epsilon} P_{[l']}(\lambda_2) P_{[k']}(\lambda_2) \prod\nolimits_0' \left(\frac{1}{\mathrm{i}\hbar} (\lambda_2 + \lambda_{[k']})^a P_{[l']}(\lambda_2 + \mathrm{i}\hbar) P_{[k']}(\lambda_2 - \mathrm{i}\hbar) \prod\nolimits_+' \right.$$
$$\left. + \frac{1}{\mathrm{i}\hbar} (\lambda_2 + \lambda_{[k']} - k\mathrm{i}\hbar)^a P_{[l']}(\lambda_2 - \mathrm{i}\hbar) P_{[k']}(\lambda_2 + \mathrm{i}\hbar) \prod\nolimits_-' \right)$$

$$B = -\frac{1}{\epsilon} P_{[l']}(\lambda_2) P_{[k']}(\lambda_2) \prod\nolimits_0' \left(\frac{1}{\mathrm{i}\hbar} (\lambda_2 + \lambda_{[k']})^a P_{[l']}(\lambda_2 + \mathrm{i}\hbar) P_{[k']}(\lambda_2 - \mathrm{i}\hbar) \prod\nolimits_+' \right.$$
$$\left. + \frac{1}{\mathrm{i}\hbar} (\lambda_2 + \lambda_{[k']} - k\mathrm{i}\hbar)^a P_{[l']}(\lambda_2 - \mathrm{i}\hbar) P_{[k']}(\lambda_2 + \mathrm{i}\hbar) \prod\nolimits_-' \right)$$

so that $A + B$ is regular.

Consider the pole at $\lambda_1 = \lambda_2 - \mathrm{i}\hbar$. Set $\lambda_1 = \lambda_2 - \mathrm{i}\hbar + \epsilon$

$$A = -\frac{1}{\mathrm{i}\hbar\epsilon} P_{[n']}(\lambda_2 - \mathrm{i}\hbar) P_{[n']}(\lambda_2) (\lambda_2 - \mathrm{i}\hbar + \lambda_{[k']})^a \prod\nolimits_0' \prod\nolimits_+'$$
$$B = \frac{1}{\mathrm{i}\hbar\epsilon} P_{[n']}(\lambda_2) P_{[n']}(\lambda_2 - \mathrm{i}\hbar) (\lambda_2 - \mathrm{i}\hbar + \lambda_{[k']} - (k-1)\mathrm{i}\hbar)^a \prod\nolimits_0' \prod\nolimits_-'$$

so that $A + B$ is proportional to

$$\sum_{[k']} \prod\nolimits_0' \left((\lambda_{[k']})^{a'} \prod\nolimits_+' - (\lambda_{[k']} - (k-1)\mathrm{i}\hbar)^{a'} \prod\nolimits_-' \right)$$

which is our identity at a lower level.

Consider the pole at $\lambda_1 = \lambda_2 + \mathrm{i}\hbar$. Set $\lambda_1 = \lambda_2 + \mathrm{i}\hbar + \epsilon$.

$$A = -\frac{1}{\mathrm{i}\hbar\epsilon} P_{[n']}(\lambda_2 + \mathrm{i}\hbar) P_{[n']}(\lambda_2) (\lambda_2 + \lambda_{[k']} - (k-1)\mathrm{i}\hbar)^a \prod\nolimits_0' \prod\nolimits_-'$$
$$B = \frac{1}{\mathrm{i}\hbar\epsilon} P_{[n']}(\lambda_2) P_{[n']}(\lambda_2 + \mathrm{i}\hbar) (\lambda_2 + \lambda_{[k']})^a \prod\nolimits_0' \prod\nolimits_+'$$

so that $A + B$ is proportional to

$$\sum_{[k']} \prod\nolimits_0' \left((\lambda_{[k']})^{a'} \prod\nolimits_+' - (\lambda_{[k']} - (k-1)\mathrm{i}\hbar)^{a'} \prod\nolimits_-' \right)$$

which is our identity at a lower level. So, if the identity holds at lower levels, our expression, as a rational function of λ_1, is regular everywhere and tends to zero at ∞, hence identically vanishes. Since the identity is true for $k = 1$ by lemma 1, it is true as stated. □

References

[1] Sklyanin E K 1985 *The Quantum Toda Chain (Lecture Notes in Physics* vol 226*)* (Berlin: Springer) pp 196–233

[2] Faddeev L and Takhtajan L 1986 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[3] Dubrovin B A, Krichever I M and Novikov S P 1990 Integrable systems I *Encyclopedia of Mathematical Sciences, Dynamical Systems IV* (Berlin: Springer) pp 173–281

[4] Sklyanin E K 1995 Separation of variables *Prog. Theor. Phys. (Suppl.)* **185** 35

[5] Krichever I M and Phong D H 1997 On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories *J. Diff. Geom.* **45** 349–89

[6] Smirnov F 1998 Structure of matrix elements in quantum Toda chain *J. Phys. A: Math. Gen.* **31** 8953–71 (*Preprint* math-ph/9805011)

[7] Babelon O and Talon M 1999 The symplectic structure of rational Lax pair systems *Phys. Lett.* A **257** 139–44 (*Preprint* solv-int/9812009)

[8] Kharchev S and Lebedev D 1999 Integral representations for the eigenfunctions of quantum periodic Toda chain *Preprint* hep-th/9910265

[9] Smirnov F A 2000 Dual Baxter equations and quantization of affine Jacobian *Preprint* math-ph/0001032

[10] Enriquez B and Rubtsov V 2001 Commuting families in skew fields and quantization of Beauville's fibration *Preprint* math.AG/0112276

[11] Smirnov F A and Zeitlin V 2002 Affine Jacobians of spectral curves and integrable models *Preprint* math-ph/0203037

[12] Babelon O and Talon M 2002 Riemann surfaces, separation of variables and classical and quantum integrability *Preprint* hep-th/0209071

[13] D'Hoker E, Krichever I M and Phong D H 2002 Seiberg–Witten theory, symplectic forms, and Hamiltonian theory of solitons *Preprint* hep-th/0212313

[14] Babelon O, Bernard D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)